Qualifying Exam. Complex Analysis. Spring 2019

Problem 1. Does there exists a function f holomorphic in the unit disk and such that

$$f\left(2^{-2n}\right) = 2^{-n}$$

for all positive integers n?

Problem 2. Let f be a holomorphic function in $\{z : \; 0 < |z-a| < 1\}$. Suppose that f has a simple pole at a. For $0 < \varepsilon < 1$, we denote by γ_ε the curve

$$\gamma_\varepsilon : [0, \pi] \to \mathbb{C}, \quad \gamma_\varepsilon(t) = a + \varepsilon e^{it}.$$

Prove that

$$\lim_{\varepsilon \to 0} \int\limits_{\gamma_\varepsilon} f(z)\, dz = \pi i \operatorname{res}_a f.$$

Problem 3. Let f be an analytic (holomorphic) function in $\mathbb{D} = \{\zeta \in \mathbb{C} : \ |\zeta| < 1\}$ such that $|f(\zeta)| < 1$, $\zeta \in \mathbb{D}$. By considering the function g defined by

$$g(\zeta) = \frac{f(\zeta) - f(0)}{1 - \overline{f(0)}f(\zeta)}, \quad |\zeta| < 1,$$

show that

$$|f(\zeta)| \leq \frac{|f(0)| + |\zeta|}{1 - |f(0)| \cdot |\zeta|}, \quad |\zeta| < 1.$$

Problem 4. Does the function

$$\sqrt{\frac{z+1}{z-1}}$$

have a holomorphic branch in $\mathbb{C} \setminus [-1, 1]$?

Problem 5. Evaluate the integral
$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}dx.$$